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NONFACETS FOR SHELLABLE SPHERES

BY

D. BARNETTE[†]

ABSTRACT

We generalize the notion of a map on a 2-sphere to maps on the n-sphere and then show that there exist combinatorial types of countries that cannot be the only type of country for a shellable n-sphere. This generalizes the well known theorem that there are no maps on the 2-sphere all of whose countries are k-gons for any $k \geqq 6$.

We shall say that a d-polytope P is *facet forming* if there exists some $(d+1)$-polytope all of whose facets are isomorphic to P. If P is not facet forming we call it a *nonfacet*. It is a well known consequence of Eulers equation that all n-gons for $n \geqq 6$ are nonfacets. This is really a topological property, however, because one cannot even draw a map on the 2-sphere in which all countries have n boundaries for $n \geqq 6$. In higher dimensions Perles and Shephard have found nonfacets [3], but their results hold only for polytopes because the nonfacets were found by considering the sum of the solid angles of the facets at various faces of a polytope.

It is rather surprising that whether a d-polytope P is a nonfacet seems to depend on the existence of large $(d-2)$-spheres in the $(d-2)$-skeleton of P. Perles and Shephard found that a d-polytope P is a nonfacet if the maximum number of k-faces of P on any $(d-2)$-sphere in the $(d-2)$-skeleton of P is at most $(d-1-k)/(d+1-k)$ times the number of k-faces of P for $0 \leqq k \leqq d-2$.

We shall derive a similar result without considering angle sums, or any other metric property. Our combinatorial approach will show the existence of nonfacets for a larger class of objects.

First, we need a generalization of the idea of a map drawn on a 2-sphere. For this we shall use *generalized combinatorial spheres* (abbreviated gcc's) which are defined as follows.

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A (-1)-gcc is the empty set, a 0-gcc is a point.

A d-gcc is a closed d-cell whose boundary is the union of a collection of k-gcc's, $-1 \leqq k \leqq d-1$, called faces of G, such that

(i) If F_1 is a face of F_2 of G then F_1 is a face of G.

(ii) If F_1 and F_2 are faces of G then $F_1 \cap F_2$ is a face of both F_1 and F_2.

Many properties of gcc's may be found in [1]. A d-gcc G is *shellable* provided its $(d-1)$-faces (hereafter to be called *facets*) can be arranged in a sequence $F_1, \cdots, F_n$ such that $\bigcup_{i=1}^{k} F_i$ is a $(d-1)$-cell for $1 \leqq k \leqq n-1$, while $\bigcup_{i=1}^{n} F_i$ is the boundary of G. While it is known that d-polytopes can be shelled [2], it is not known if any unshellable spheres exist.

THEOREM. *If C is a shellable d-gcc then some facet F of C has a $(d-2)$-sphere in its $(d-2)$-skeleton, containing more than one third of the $(d-2)$-faces of F.*

PROOF. Let C be shelled in the sequence $F_1, \cdots, F_n$. Let $E_k = \bigcup_{i=1}^{k} F_i$. Let $f_0(E_k)$ be the number of $(d-2)$-faces on the boundary (in C) of E_k that belong to exactly one of $F_1, \cdots, F_k$ and let $f_1(E_k)$ be the number of $(d-2)$-faces on the boundary of E_k that belong to more than one of $F_1, \cdots, F_k$. We now look at how f_0 and f_1 change when we add F_{k+1} to $F_1, \cdots, F_k$. Let $C_1 = F_{k+1} \cap (\bigcup_{i=1}^{k} F_i)$. Since the sequence of F's gives a shelling we see that C_1 is a closed $(d-1)$-cell and its complement in the relative boundary of F_{k+1} is a set whose closure we shall call C_2. The sets C_1 and C_2 intersect on a $(d-2)$-sphere S. Let n_1 be the number of $(d-2)$-faces of F_{k+1} in C_1 but not lying in S, let n_2 be the number of $(d-2)$-faces of F_{k+1} in C_2 but not in S and let n_3 be the number of $(d-2)$-faces in S. Finally let n_4 be the number of $(d-2)$-faces of S that belong to more than one of $F_1, \cdots, F_k$. We have that

$$f_0(E_{k+1}) = f_0(E_k) + n_2 - n_3 + n_4,$$

$$f_1(E_{k+1}) = f_1(E_k) + n_3 - n_4 - n_1.$$

We define $X(E_i) = f_0(E_i) - f_1(E_i)$ thus $X(E_{k+1}) = X(E_k) + n_1 + n_2 - 2n_3 + 2n_4$. We observe that $X(E_1) > 0$ and $X(E_{n-1}) < 0$, thus at some step X must decrease when some facet F_{k+1} is added. Thus we must have $n_1 + n_2 - 2n_3 < 0$ at this step. Thus if N is the number of $(d-2)$-faces of F_{k+1} then we have $N - 3n_3 < 0$ or $n_3 > N/3$, that is, more than one third of the $(d-2)$-faces of F_{k+1} are on S.

COROLLARY. *If C is a d-gcc such that every $(d-2)$-sphere in the $(d-2)$-skeleton of C has at most one third of the $(d-2)$-faces of C then C is a nonfacet for shellable $(d+1)$-spheres, that is, no shellable $(d+1)$-sphere has all of its facets isomorphic to C.*

REMARKS. (1) This theorem may give more insight into why large $(d-2)$-spheres are necessary in the $(d-2)$-skeletons of facet forming spheres. The reason is that as one tries to build up the boundary of a shellable sphere with such facets, the cells that one builds will not "close up" if one can't get enough $(d-2)$-faces on the boundary of the intersection of the next facet with the cell that has been built. If one tries this with hexagons in the plane one can actually observe this failure to close up, because the cells one builds always have too many 2-valent vertices on the boundary.

(2) Our theorem is a generalization of Perles and Shephard's theorem for the case $k = d-2$. Our theorem can be proved exactly the same way for faces of dimensions less than $d-2$ and we get

THEOREM 2. *If C is facet forming for shellable d-gcc's, then some $(d-2)$-sphere in the $(d-2)$-skeleton of C contains more than one third of the k-faces of C for $0 \leqq k \leqq d-2$.*

Perles and Shephard's theorem is stronger than ours for $k < d-2$. The author would like to know why this happens.

(3) The author has the feeling that in considering $f_0(E_i) - f_1(E_i)$ he is really looking at a special case of a much more general argument, but it is not clear what this more general argument would be.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIF. 95616 USA